Indian Statistical Institute, Bangalore
B. Math (Hons.) Second Year
First Semester - Group Theory

Midterm Exam — Date: 13th September 2024
Maximum marks: 30 — Duration: 2 hours

Answer any five, each question carries 6 marks, total marks: 30

1. (a) Suppose a group G acts on a set X. For $x, y \in X$, prove that $Gx = Gy$ or $Gx \cap Gy = \emptyset$ and X is a disjoint union of orbits.

   (b) State and prove Lagrange's Theorem *(Marks: 4)*.

2. (a) Prove that any group of order n is isomorphic to a subgroup of $GL_n(\mathbb{R})$.

   (b) Prove normal subgroups of order 2 are contained in the center *(Marks: 2)*.

3. State and prove orbit-stablizer theorem and first Sylow theorem.

4. (a) Prove that any subgroup H of a finite group G with $o(H) = p^k$ $(k > 0)$ is contained in a Sylow p-subgroup of G *(Marks: 4)*.

   (b) Prove that any finite abelian group is a product of its Sylow subgroups.

5. (a) If σ and τ are permutations which act on disjoint set of indices. Prove that $\sigma\tau = \tau\sigma$.

   (b) Prove that nontrivial permutations can be uniquely expressed as a product of cyclic permutations which act on disjoint sets of indices *(Marks: 3)*.

6. (a) Find $Z(\sigma)$ where $\sigma = (1, 2, \cdots, r) \in S_n$. *(Marks: 3)*.

   (b) Find $\{\sigma(1, 2, 3)\sigma^{-1} \mid \sigma \in A_n\}$ for $n \geq 5$.

7. (a) Prove $\mathbb{Q}$ can't be written as a product of its proper subgroups *(Marks: 3)*.

   (b) Describe the center of $A \ltimes_\phi B$.